Lp theory for outer measures

X : metric space

All sets are assumed to be Borel and all functions from $\mathcal{B}(X) = \{\text{Borel measurable fns}\}$

Aim: To construct quasi-norms (semi) $L^p(X,\sigma,S)$ and $L^{p,\infty}(X,\sigma,S)$ on subspaces of $\mathcal{B}(X)$, and to apply these in harmonic analysis.

The basic structures:

① **Defn**: A premeasure is function $\sigma : \mathbb{E} \to [0,\infty)$ defined on a collection $\mathbb{E}$ of subsets of X.
(No assumptions on σ !)

② σ generates an outer measure ($\mu : 2^X \to [0,\infty]$) on all subsets of X:

$$\mu(E) := \inf \sum_{E' \in \mathbb{E}'} \sigma(E') \quad , \quad E \subset X$$

(over all countable $\mathbb{E}' \subset \mathbb{E}$ s.t. $E \subset \bigcup_{E' \in \mathbb{E}'} E'$)

For any σ, μ will be an outer measure:

(1) $E \subset E' \Rightarrow \mu(E) \le \mu(E')$

(2) $\mu(\emptyset) = 0$

(3) $\{E_j\}$ countable $\Rightarrow \mu(\bigcup E_j) \le \sum \mu(E_j)$

↳ Pf: $E_j \subset \bigcup_i F_j^i \quad , \quad F_j^i \in \mathbb{E}$

$$\sum_i \sigma(F_j^i) \le \mu(E_j) + \varepsilon 2^{-j}$$

$$\Rightarrow \underbrace{\sum_{ij} \sigma(F_j^i)}_{\ge \mu(\bigcup_j E_j)} \le \sum_j \mu(E_j) + \varepsilon$$

(ε arbitrary >0) /

Note: In general $\mu|_{\mathbb{E}} \neq \sigma$

3. **Defn. 2.3**: A *size* is a map $S : B(X) \times \mathbb{E} \to [0, \infty]$

satisfying:

(1) $|f| \le |g| \Rightarrow S(f, E) \le S(g, E)$

(2) $\lambda \in \mathbb{C} \Rightarrow S(\lambda f, E) = |\lambda| \, S(f, E)$

(3) $\exists C < \infty \; \forall f, g, E$:

$$S(f+g, E) \le C \left(S(f, E) + S(g, E) \right)$$

(Often $C = 1 \iff S =$ seminorm)

Example 1 (Lebesgue): $X := \mathbb{R}^n$, $\mathbb{E} := \{\text{dyadic cubes}\}$ (alternatively balls)

$\sigma(Q) := 2^{nk}$ if sidelength $l(Q) = 2^k$

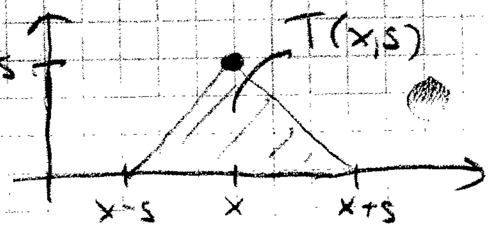
$$S(f, Q) := \frac{1}{\sigma(Q)} \int_Q |f(x)| \, dx \quad \left(\text{or } S_p(f, Q) = \left(\frac{1}{\sigma(Q)} \int |f|^p dx \right)^{1/p} \right)$$

Here the outer measure μ restricts to Lebesgue measure on Leb. meas. sets.

Example 2 (Carleson): $X := \mathbb{R}^2_+ = \mathbb{R} \times (0, \infty)$

$\mathbb{E} := \{\text{tents } T(x, s) \; ; \; (x, s) \in \mathbb{R}^2_+\}$

$\sigma(T(x, s)) := s$



$$S(f, T(x,s)) := \underbrace{\frac{1}{s}}_{= \sigma(T)} \iint_{T(x,s)} |f(y, t)| \underbrace{\frac{dx\,dt}{t}}_{\substack{\text{natural measure} \\ \text{on } \mathbb{R}^2_+}}$$

or more generally

$$S_p(f, T) = \left(\frac{1}{\sigma(T)} \iint_T |f|^p \frac{dx\,dt}{t} \right)^{1/p} \qquad 1 \le p < \infty$$

$$S_\infty(f, T) = \lim_{p \to \infty} S_p(f, T) = \sup_T |f|$$

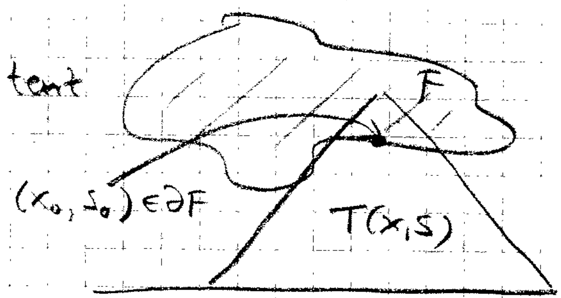
Claim: Only $\emptyset$ and X are measurable w.r.t. μ in Ex 2. 1:3

Pf: Assume $F \subset X$ is μ-measurable, meaning

$\forall E \in \mathbb{E}$: $\mu(E \cap F) + \mu(E \cap F^c) \underset{(=)}{\leq} \mu(E)$

$(\Rightarrow \forall E \subset X)$

Consider $(x_0, s_0) \in \partial F$ and tent $T(x,s) \ni (x_0, s_0)$.



$$\Rightarrow \underset{\sigma}{\overset{\shortparallel}{\mu}}(T(x,s)) = s$$

$$< \underbrace{\mu(T(x,s) \cap F)}_{\approx s} + \underbrace{\mu(T(x,s) \cap F^c)}_{\approx s}$$

since $G \ni (y,t) \Rightarrow \mu(G) \geq t$, as least one tent with height $> t$ is needed to cover G.

4 Norms derived from σ and S:

A size S (and implicitly the collection $\mathbb{E}$) gives a supremum-notion:

out $\sup_F S(f)$ = the outer supremum of $f \in \mathcal{B}(X)$ on $F \subset X$

$$:= \sup_{E \in \mathbb{E}} S(f \cdot 1_F, E)$$

Ex 1: out $\sup_F S_p(f) = \overset{\text{ess sup}}{\sup_F} |f|$

by Leb. diff. thm. $\forall p$

Ex 2: out $\sup_F S(f) \neq \sup_F |f|$

Assume e.g. $\operatorname{supp} f \subset \mathbb{R} \times (1, \infty)$ and $\iint_{\mathbb{R}^2_+} |f| \frac{dx\,dt}{t} = C < \infty$ but f unbounded.

Then out $\sup_{\mathbb{R}^2_+} S(f) \leq \frac{1}{1} \cdot C < \infty$.

The crucial definition for outer L^p spaces (1:4)

Defn 2.5 $f \in \mathcal{B}(X)$, $\lambda > 0$. Then we def. the super level measure as

$$\mu(S(f) > \lambda) := \inf \{ \mu(F) ; \text{outsup}_{X \setminus F} S(f) \leq \lambda \}.$$

[Note:
$$\mu(S(f) > \lambda) \neq \mu(\{x ; |f| > \lambda\}) \;]$$

We define strong and weak outer L^p (quasi-) norms:

$$\begin{cases} \|f\|_{L^p(X,\sigma,S)} := \left(\int_0^\infty p \lambda^{p-1} \mu(S(f) > \lambda) \, d\lambda \right)^{1/p}, & 0 < p < \infty \\ \|f\|_{L^\infty(X,\sigma,S)} := \text{outsup}_X S(f) \end{cases}$$

$$\|f\|_{L^{p,\infty}(X,\sigma,S)} := \left(\sup_{\lambda > 0} \lambda^p \mu(S(f) > \lambda) \right)^{1/p}, \quad 0 < p < \infty$$

Let $\|\cdot\|_y$ be any of these "norms". Then:

(1) $|f| \leq |g| \Rightarrow \|f\|_y \leq \|g\|_y$

(2) $\|\lambda f\|_y = |\lambda| \, \|f\|_y$

$\hookrightarrow$ $\mu(S(\lambda' f) > \lambda) = \inf \{ \mu(F) ; \underbrace{\sup_E S(\lambda' f 1_{X \setminus F}, E) \leq \lambda}_{\Leftrightarrow \sup S(f 1_{X \setminus F}, E) \leq \lambda/\lambda'} \}$
$= \mu(S(f) > \lambda/\lambda')$

(3) $\exists C < \infty \; \forall f, g$:
$$\|f + g\|_y \leq \|f\|_y + \|g\|_y.$$

$\hookrightarrow$ Details: For super level measures

Let $\varepsilon > 0$. Take $F_f, F_g \subset X$ s.t.

$\sup_E S(f 1_{X \setminus F_f}, E) \leq \lambda$, $\mu(F_f) \leq \mu(S(f) > \lambda) + \varepsilon$

$\sup_E S(g 1_{X \setminus F_g}, E) \leq \lambda$, $\mu(F_g) \leq \mu(S(g) > \lambda) + \varepsilon$

Then for $F := F_f \cup F_g$:

$$\sup_E S((f+g) 1_{X \setminus F}, E) \overset{\text{from defn. 2.3}}{\leq} C \big(\overbrace{S(f 1_{X \setminus F}, E)}^{\leq f 1_{X \setminus F_f}} + \overbrace{S(g 1_{X \setminus F}, E)}^{\leq g 1_{X \setminus F_g}} \big)$$

1.5

$\leq 2C\lambda$

$\Rightarrow \mu(S(f+g) > 2C\lambda) \leq \mu(F) \leq \mu(F_f) + \mu(F_g)$

$\leq \mu(S(f) > \lambda) + \mu(S(g) > \lambda) + \underbrace{2\varepsilon}_{\text{arbitrary} > 0.}$

<u>Note</u>: Even strong onto L^p, $p > 1$, may be only a quasi-norm. (Even if $C = 1$ for size.)